When the Old Meets the New: Examples of What Established Analytical Methods Look Like in a Modern Computer Environment

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Abstract

There is a vast number of analytical solutions of Ordinary and Partial Differential Equations (ODE and PDE) available in engineering literature, books, journals and teaching material, starting from the definition of the Partial Differential Equation (PDE) for thin plates formulated by Lagrange in 1811 /1/ see figure 1, the mathematical breakthroughs by Augustin-Louis Cauchy in early 1800s /2/ and Claude-Louis Navier /3/ using double Fourier series to solve the problem of a simply supported plate with different types of loads in 1820. The focus in this paper is the analytical solution of rectangular plates.

To solve these PDEs by hand calculations were the norm, limiting the practical use of these mathematical findings significantly. Today's engineers and designers working in product development have vast computer resources available to them to implement these PDEs for better understanding of the behaviour of rectangular plates. The introduction of Formulation, Validation and Verification in product development has actualised the analytical solutions, as numerical solutions computed using FEA technology must be compared against "exact solutions" for verification.

It is pertinent to ask the question: "What would the forefathers in Classic Solid Mechanics have done if they had our computer resources available to them?" A number of examples are made to show what effective use of state-of-the-art computing can do to revive the classical methods.

1. Introduction

Mathematicians working in the 19th century saw Classic Solid Mechanics as a rich source of unsolved problems to address. Many of the familiar names famous for their contributions in other domains of mathematics have also contributed in this domain /4/. In the 20th century engineers and mathematicians combined their interests in solving practical problems using the same approach, and with increased understanding of the underlying mathematical theory further refinements and improvements were made to create more advances PDEs. Even in the mid-1950s the tools available to solve these equations were limited to logarithm tables, slide rules and handheld calculators.

The Classical Solid Mechanics literature is focused on defining the theory for PDEs, to describe the solution methods and outline the resulting equations for deformations and bending moments. Solving the PDE for Classical Plate Theory (CPT) for practical load cases and boundary conditions were laborious and time consuming, hence the literature is full of excellent theory, but few examples of solutions to them, making it nearly possible for practicing engineers to make use of the theory.

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w\, \partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \qquad (1)$$

or

$$\nabla^4 w = \frac{q}{D} \qquad (2)$$

Where

w is the transverse displacement

q is the load on the plate expressed as q =f(x,y)

D is the flexural rigidity of the plate

$$q = \mathrm{f(x, y)}$$

$$D = \frac{Eh^3}{12(1-\nu^2)}$$

Figure 1: The Classical Plate Theory (CPT) Partial Differential Equation (PDE)

Figure 1 describes the bi-harmonic forth-order PDE governing the behaviour of thin rectangular plates known as the Classic Plate Theory (CPT). The PDE combines the three components that make up any PDE for rectangular plates:

- Force Resultants-Stress relationships - No shear-effect included
- Stress-Strain relationships - Hooke's Law for isotropic materials
- Strain-Displacement relationships - Small deformations

Together these choices for the formulation are known as the Kirchhoff-Love formulation for thin plates /5/ in the FEA context called thin-plate theory.

The lack of effective calculation methods resulted in a narrowing of the practical use of the theory as shown in figure 2. Typically, from /6/:

- Equation (3) shows the Classic Plate Theory PDE, which represents any rectangular plate according to Kirchhoff-Love thin plate theory.
- Equation (4) shows the general solution found using the Navier's method for Simply Supported Rectangular Plates using double Fourier series to represent the displacement and load, the general expression for the deflection w is given for any load type.
- Equation (5) shows the expression of a_{mn} specialised for a uniformly distributed load over the entire surface of the plate.
- Equation (6) shows the expression for the maximum deflection in the centre of any plate
- Equation (7) shows the approximate value for the centre of a square plate.

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^2 w\,\partial^2 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D} \tag{3}$$

$$w = \frac{1}{\pi^4 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{a_{mn}}{\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \tag{4}$$

$$a_{mn} = \frac{4q_O}{ab}\int_0^a\int_0^b \sin\frac{m\pi x}{a}\sin\frac{n\pi y}{b}\,dxdy = \frac{16q_O}{\pi^2 mn} \tag{5}$$

$$w_{max} = \frac{16qo}{\pi^6 D}\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{(-1)^{\frac{m+n}{2}-1}}{mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \tag{6}$$

$$w_{max} = \frac{4q_O a^4}{\pi^6 D} = 0.00416\frac{q_O a^4}{D} \tag{7}$$

Figure 2: From the general to the specific, showing diminishing generality, from /6/

This rapid transformation from the general solution to a particular case is concluded by the statement: “This is a rapidly converging series and satisfactory approximation is obtained by taking only the first term of the series, … This result is about 2½ per cent in error”.

The example here is typical for many books on classical solid mechanics. Even modern treatises place scant emphasis on practical use of the often-elaborate theory developed in the early chapters. As the theory develops, the simplest cases are chosen, and most others are conveniently discarded. The examples are chosen to make it simple to calculate the deformation w, leaving the user to develop the mathematical expressions for all the other result components needed to solve the full breadth of cases. The reasons for this shortcoming include the need for long-winded manual calculations combined with the lack of effective means of carrying them out.

2. From manual calculations to the first computer results

To solve the PDE using Fourier series is covered well in available literature. To develop the resulting equations, require substantial theoretical work and to produce non-dimension tables for the use by practicing engineers in their daily work even more manual effort. A few examples from the literature on plates and shells are included here to show the effort involved and the transition into the computer age for classical solid mechanics.

Examples are chosen at random, there may be other more suitable ones:
- Ernst Bittner: "Platten und Behalter" /7/ an impressive compendium of solutions to the Classical Plate Theory (CPT) equations with Navier's and Lévy's methods for a large number of boundary condition permutations and load types. The solutions to these equations are presented in 217 figures and 284 tables as a tool for engineers working in reinforced concrete plate designs.
- Rűdiger-Urban: KreisZylinderscalen (Circular Cylindrical Shells) was published in 1955 /8/ with the theoretical background for the 8th order PDE governing circular cylindrical shells and is a fantastic collection of non-dimensional tables, all calculated and checked by a team of mathematicians at the Calculating Office at the Institute for Applied Mathematics at the Dresden Technical University, Germany. A number of parameters are systematically changed to tabulate the response for deformations and internal forces. The Preface ends: "May the tables win many friends. Dresden, in the autumn 1955". This was before the age of computing.

The introduction of the early computers was welcomed by a new generation of engineers / mathematicians who could now solve complicated PDEs using algorithms written initially in machine-code to create new and more comprehensive tables. Examples include
- Prof. Dr Tech Ivar Holand, professor at Department of Structural Mechanics at the Norwegian Institute of Technology in Trondheim, Norway, who used manual calculation methods to solve the 4th order PDE for cylindrical tanks with linearly variable thickness in the late 1950s. The solution of the PDE included Kelvin functions /9/ which requires substantial manual effort for the calculations. In 1960 he used Ferranti Mercury (production no: 001) computer to compute the deformations and bending moments for the PDEs, resulting in a number of tables with non-dimensional values for practical use in cylindrical tubes, water tank and arc-dam design /10/.
- Prof. J. E. Gibson who started off with hand-held calculators for calculation and checking of the tables for "The Design of Cylindrical Shell Roofs" in his first edition in 1954 /11/ and added chapters on the programming of the Manchester Mark 1 computer in "autocode" to solve the same equations in the second edition from 1960 /12/. He further pioneered the use of the digital computer in "Computing in Structural Engineering" /13/ outlining the components that make up a modern computer like the Mercury and Atlas computers at the University of Manchester, the I.C.L. 1905 and the CDC 7600 mainframes in London, UK. He explains further: "More recently the author has

been actively involved in the use of a mini computer, the PAC 16, and an electronic calculator, the HP 35, both of which have influenced the presentation of this book”. He included a short course in FORTRAN (FORmula TRANslation) /14/ in the book together with the programs he used himself to solve the 8th order PDE for cylindrical shell roofs and other shell structures. J. E. Gibson included no tables, but any reader would be free to implement his simple but effective Fortran code.

Since then the use of the analytical methods have faded somewhat, replaced by approximate methods based on alternative solution methods for the same PDEs, using finite element analysis (FEA) for plates and shells.

Little has been done to revive the value of this material described in detail in the books listed and many others, including **“Theory of Plate and Shells” by Stephen P. Timoshenko and S. Woinowsky-Krieger /**15/ and **“Stresses in Shells” by William Flűgge** /16/ both classics found in the shelves of seasoned structural engineers. The reasons are obvious, as visualised in figure 3:

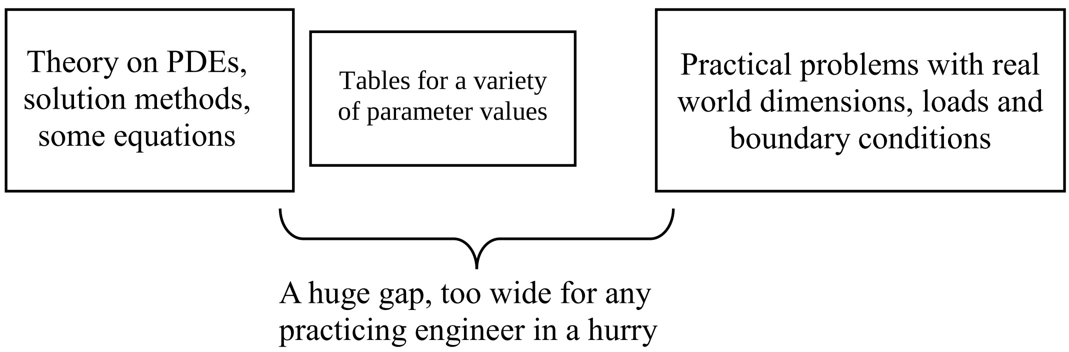


Figure 3: The distance between the theory of CPT’s PDE and practical use, a big void for any engineer working in a project with budget and time boundaries

Practical engineering problems are often very different from the theoretical material in the literature and bridging the gap between the theory and practice requires too much time and effort to be feasible. Non-dimensional tables are of limited or no use as they cover simple problems and list variables and their values different form the practical problems an engineer is grappling with.

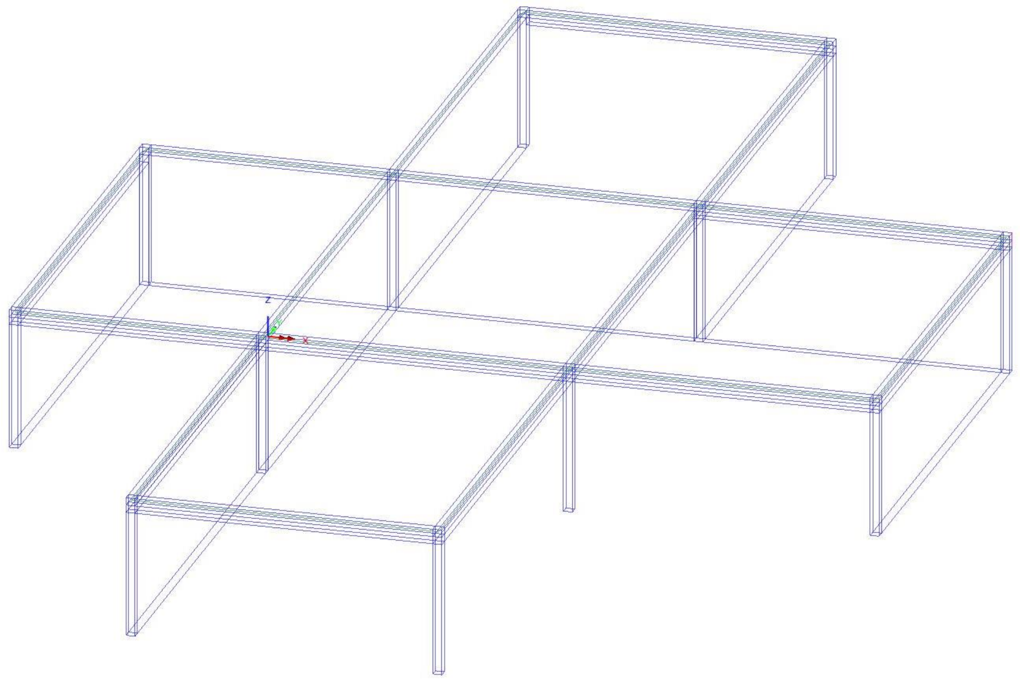


Figure 4: A practical problem, beyond what is in the classical literature

Here is the dilemma: the practical problem that an analyst wants answers to is not like the solutions that the books have partly solved. Typically, a plate may be rectangular, but it rests on elastic beams and concrete walls at the boundaries and have continuous connections to the plates beyond its edges as in figure 4.
It is a reinforced concrete plate, hence should be modelled as a medium thick orthotropic plate where a uniformly distributed load is a particular load case. Other load cases for critical load may be combinations of patch loads and point loads where the combinations of loads on the plate and its adjoining plates have to be computed. The results from the analyses will subsequently be used for Serviceability Limit State (SLS) and Ultimate Limit State (ULS) calculations.

Figure 5 shows the centre plate to be analysed with the idealisation offset. The behaviour of the plate is determined by the adjoining plates, the supporting beams and walls. The boundary conditions for the plate edges can be choses as indicated, for the edge with a supporting wall: Continuous Plate With Supported Edge and Free Rotation, and for the edges with a supporting beam: Elastically Built-in Edge. These terms are detailed in table 2. This case is not explicitly included in /7/ so a project engineer is encouraged to develop the solution for this particular problem and follow the well-documented and detailed structure of the many worked plate solutions.

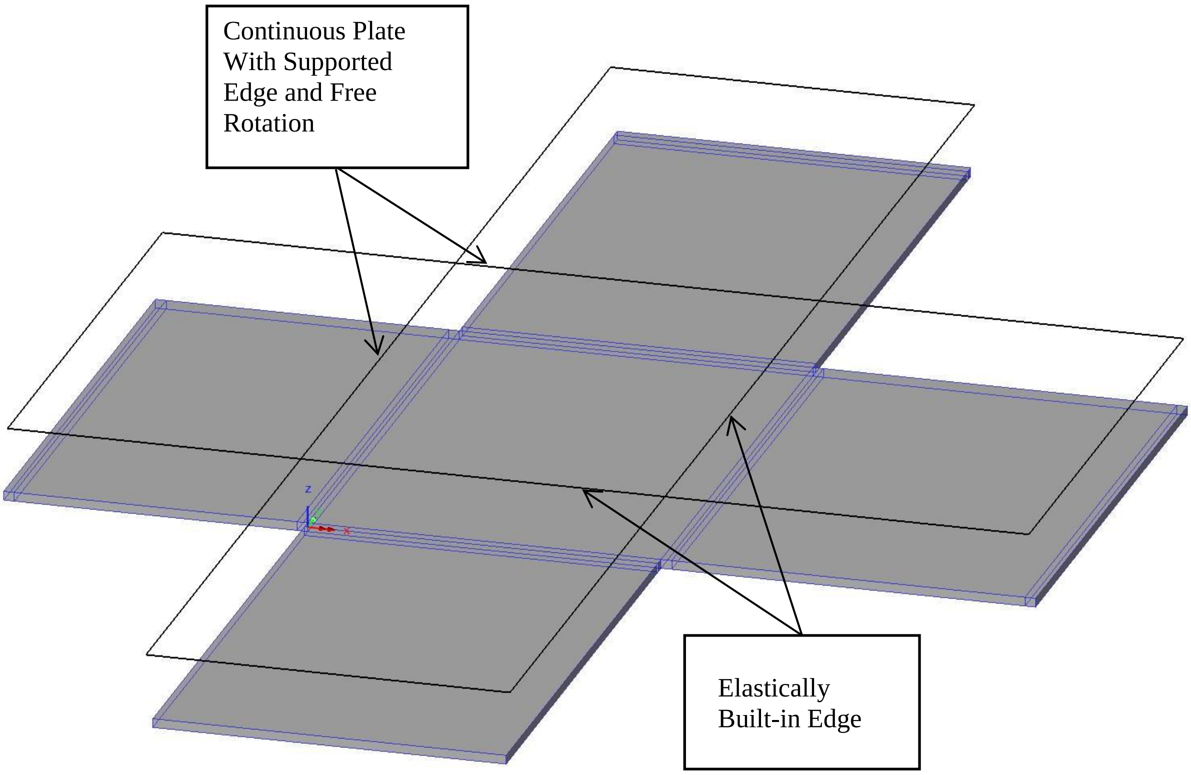


Figure 5: a practical problem, the plates and their idealisations

3. Formulation, Verification and Validation

A rich source of analytical solutions is needed for comparison to the numerical solutions produced in modern computer systems for mathematical modelling. A thorough understanding of and practice in the use of Formulation, Verification and Validation technology is the basis for achieving the necessary accuracy in approximate simulations using FEA running commercial analysis codes /17/, /18/, /19/.

All analysis problems can be assumed to be non-linear at least in part. The experienced analyst must make decisions on how the non-linear behaviour of his real-world problem can be reasonably represented as linear. Linear assumptions are used very successfully throughout industry together with dimensional reduction from 3D elasticity to 2D plates and shell formulations. There are a number of alternative PDEs for rectangular plates to choose from, it is not given up-front which is better at representing the real-world problem at hand.

4. Formulation

The formulation of the mathematical model is a matter of choice among a large number of PDEs. The analyst must make a qualified choice of the representation of their problem at hand to ensure that the verification and validation processes end with approval. For rectangular plates, there are a vast number of alternative formulations, all making assumptions about the stresses and strains over the thickness of the plate to reduce the 3D elasticity problem to a two-dimensional one.

Figure 6 shows a rectangular plate with a uniformly distributed load shown as a block above the plate and with the mid-plane marked. The dimensions for the plate is l_x = 10000mm and l_y = 15000mm. Three alternative thicknesses are shown from top to bottom:
- h = 2000, which gives a thickness to min(l_x,l_y) ratio of 1/5
- h = 1000 which gives thickness to min(l_x,l_y) ratio of 1/10
- h = 200 which gives thickness to min(l_x,l_y) ratio of 1/50

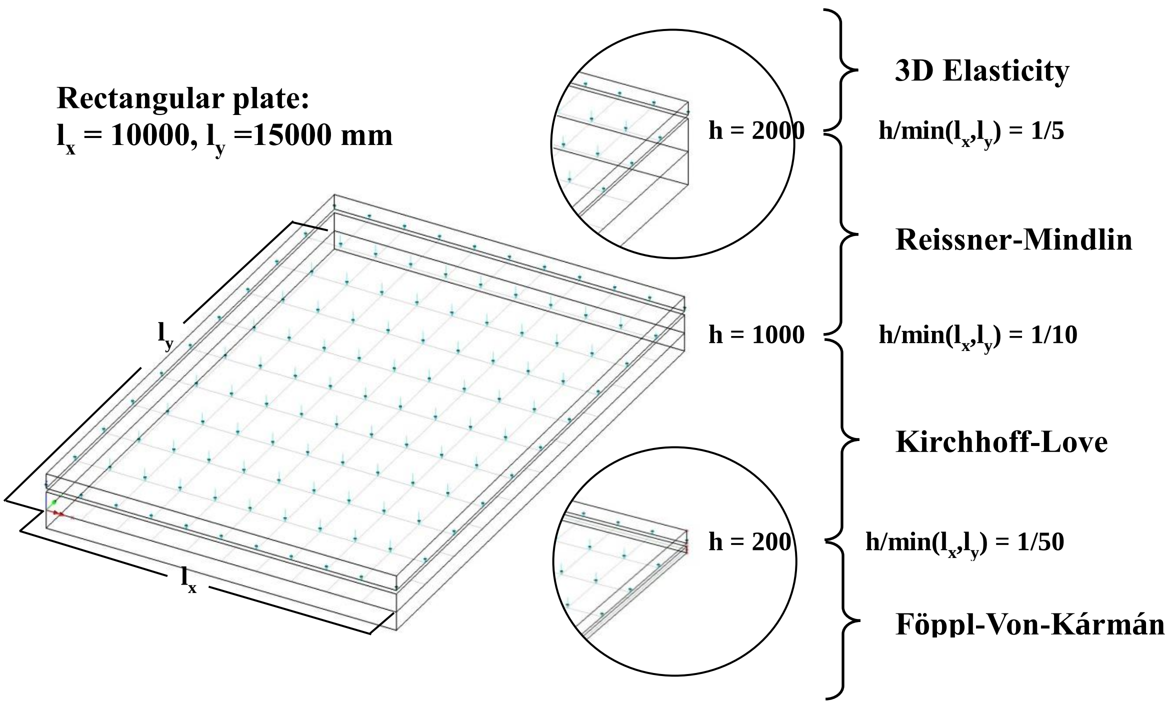


Figure 6: Alternative Formulations for rectangular plates

Kirchhoff-Love plate theory has a validity range for deformations smaller than 1/20 of the thickness of the plate.
Föppl-von Kármán plate theory has a validity range for geometric non-linearity where the deformations are in the same range as the thickness of the plate. For larger deformations that also cause the material to behave beyond Hooke’s law, other non-linear theories apply.

Rectangular Plate	Moderately Thick	Thin	Very Thin
Min(h/l_x, h/l_y)	1/5 to 1/10	1/10 to 1/50	< 1/50
Formulation	Transverse shear deformation included	Without Transverse shear deformation, popular for practical applications	Geometrically non-linear with membrane deformation
Plate Theory	Reissner-Mindlin	Kirchhoff-Love	Föppl-Von Karman
Related Beam Theory	Timoshenko	Euler-Bernoulli	Theory of 2^{nd} order

Table 1: shows a separation of the alternative plate theories based on the thickness of the plate, developed from the table in /20/.

The boundaries between the different theories and subsequent choice of PDEs are blurred. As an engineer, I have to probe:
- when is a 3D elasticity problem representing a thick plate better represented as a moderately thick plate using the 2D simplification of Reissner-Mindlin formulation?
- when does a moderately thin plate no longer exhibiting the effect of shear, hence is better represented using thin plate theory based on Kirchhoff-Love formulation?
- when is a thin plate so thin that membrane forces become dominant and the Föppl-von Kármán's formulation better represents the geometric non-linearity in a plate?

There are also other plate bending theories in the literature where examples show better performance than the established formulations listed here /5/. Typically, in this paper all Formulations are given double names. For the purist this ignores the fact that there are differences between the Formulations developed by the two in the chosen name combination as well. As a consequence, the same underlying axioms for a Formulation may be given different names across the literature on plates.

5. Verification

The verification of the analyses is based on access to an exact solution to compare the approximate solution achieved using FEA. Typically, the criteria for verification use a measure of relative error to be within prescribed tolerances as shown in figure 7, taken from /17/.

The u_{EX} and u_{FE} should not be influenced by each other, i.e., they should have different origins. The exact solution should ideally be an analytical solution to the problem studied. However, the number of analytical solutions is limited, and their formulation may not match the problem in hand. In general, an analytical solution doesn't exist. Alternatively, a high-resolution FE analysis can be used as a substitute /19/. Measured data could be used if a suitable data set is available.

To verify the accuracy of a chosen formulation, well established working practices for numerical analyses exist, like convergence studies using alternative densities in an FEA mesh /21/.

$$\frac{\phi_i(u_{EX}) - \phi_i(uFE)}{\phi_i(u_{EX})} \leq \tau_i \qquad (8)$$

Where

- $\phi_i(u_{EX})$ is the exact solution for a particular result of interest

- $\phi_i(u_{FE})$ is the approximate solution from a numerical analysis for a particular result of interest

- i is an index for the number of results of interest for consideration, for example maximum displacement, temperature, stress, etc.

-τ_i is the relative error for the FE analysis compared with the EXact solution

Figure 7: a measure of relative error for evaluating the accuracy of the simulations

However, for analytical solution methods the functions representing the behaviour of a rectangular plate are continuous functions and the calculated result components are accurate for however dense point-grid is chosen.

For the Navier's and Lévy's solution methods accuracy is a function of the Fourier series, i.e., how many terms should be included in the calculations to achieve sufficient accuracy and how fast do the expressions for the individual result component converge. This was a major concern before computers were used to calculate the results components. These days, this is no longer a problem.

The requirement for an exact solution in the Verification process justifies any effort to revive the Classic Solid Mechanic theory, solution methods and practical solutions.

The i in Figure 7 implies the existence of a number of results components that are of interest for understanding of the plate behaviour and for dimensioning in design. The CPT PDE creates a solution that is expressed as the transverse deformation of the plate, w as shown in equation 4. The bending moments and shear forces are expressed as derivatives of w. The total number of result components is ten when the deformation is included. These can be computed and visualised for increased understanding of the behaviour of plates. More importantly, the different result components are used to define the boundary conditions for solving the CPT for a number of practical problems.

Table 2 is taken from /7/ where the six alternative boundary conditions are subsequently used for the development of specific solutions for the CPT. Most other literature sources limit their exploration to the first three boundary conditions in this table: Simply Supported, Built-in and Free Edges and develop expressions and tables for any permutations of these.

Practical engineering problems are often continuous plates with different kinds of support and their boundary conditions are more like the two at the bottom of the table: Continuous Plate with Supported Edge and Free Rotation, and Elastically Built-in Edge, see figures 5 and 6.

In /6/ the last two boundary conditions are detailed further, combining the deformation and rotations with that of an elastic beam, combining Kirchhoff-Love plate theory with the equivalent Euler-Bernoulli elastic beam theory including torsion, see /22/. The theory is not fully developed, and these boundary conditions are not used when the theory for continuous plates are subsequently developed, see /23/. There is more work to do.

The Navier's method can only be used for rectangular plate with all four edges simply supported. This lack of versatility is eased by using the superposition principle where the effect of bending moment at either of the boundaries can be added to the Navier's solution and elastically built in and fully built-in edges can be represented.
The list of boundary conditions in table 2 requires the Lévy's method of single Fourier series where two opposite edges are simply supported and the other two have any of the listed boundary conditions. Superposition of edge moments extends the range of plates that can be calculated using Lévy's method. Each permutation of boundary conditions results in a particular expression for the displacement w, hence the comprehensive coverage of alternative plates in books like /7/.

All ten result components derived from the w in equation 9 are used in one or several of the mathematical expressions for the boundary conditions listed in table 2.

How does the representation of result components, load types and boundary conditions compare between analytical and numerical solution methods?

The list of ten result components must also be output in numerical solution methods. However, not all of them are part of the standard output from FEA analyses.

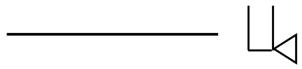
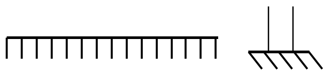
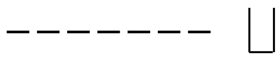
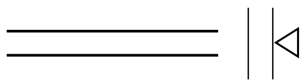
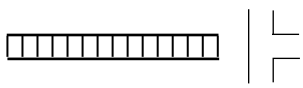
Boundary Condition	Mathematical	Graphical
Simply Supported Edge	$w = 0$ $\quad \frac{\partial^2 w}{\partial x^2} = 0$ for x=0 and $w = 0$ $\quad \frac{\partial^2 w}{\partial y^2} = 0$ for y=0 and	
Fixed or Built-in	$w = 0$ $\quad \frac{\partial w}{\partial x} = 0$ for x=0 and $w = 0$ $\quad \frac{\partial w}{\partial y} = 0$ for y=0 and	
Free Edge	$\frac{\partial^2 w}{\partial x^2} = 0$ $\ [\frac{\partial^3 w}{\partial x^3} + (2\text{-}\nu)\frac{\partial^3 w}{\partial x \partial y^2}\ ] =$ for x=0 and $\frac{\partial^2 w}{\partial y^2} = 0$ $\ [\frac{\partial^3 w}{\partial y^3} + (2\text{-}\nu)\frac{\partial^3 w}{\partial x^2 \partial y}\ ] =$ for y=0 and	
Sliding Edge With Fixed	$\frac{\partial w}{\partial x} = 0$ $\quad [\frac{\partial^3 w}{\partial x^3} + \frac{\partial^3 w}{\partial x \partial y^2}\ ] =$ for x=0 and $\frac{\partial w}{\partial y} = 0$ $\quad [\frac{\partial^3 w}{\partial y^3} + \frac{\partial^3 w}{\partial x^2 \partial y}\ ] =$ for y=0 and	
Continuous Plate With Supported Edge and Free	$w = 0$ $\quad \frac{\partial w}{\partial xleft} = \frac{\partial w}{\partial x_{right}}$ for x=0 and $w = 0$ $\quad \frac{\partial w}{\partial y_{left}} = \frac{\partial w}{\partial y_{right}}$ for y=0 and	
Elastically Built-in	$w = 0$ $\quad \frac{\partial w}{\partial x_{plate}} = \frac{\partial w}{\partial x_{support}}$ for x=0 and $w = 0$ $\quad \frac{\partial w}{\partial y_{plate}} = \frac{\partial w}{\partial y_{support}}$ for y=0 and	

Table 2: The full set of boundary conditions for rectangular plates

The alternative boundary conditions in table 2 must have a corresponding definition for numerical methods so direct comparisons can be made. In the same vain, the set of load types must span the same range of alternatives for both analytical and numerical methods.

For numerical methods like FEA convergence studies are a key to end up with reliable results. Literature on FEA methods and Formulation, Validation and Verification contains the theoretical base for convergence studies, but the technology for effective computations is lacking. There is work to do.

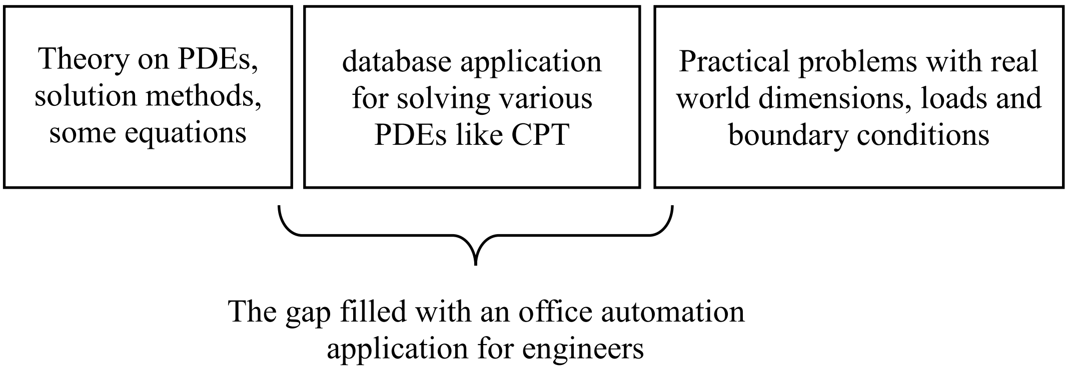


Figure 8: A modern link between the established analytical theory and practical engineering problems

6. Design of a modern computer system for solving PDEs

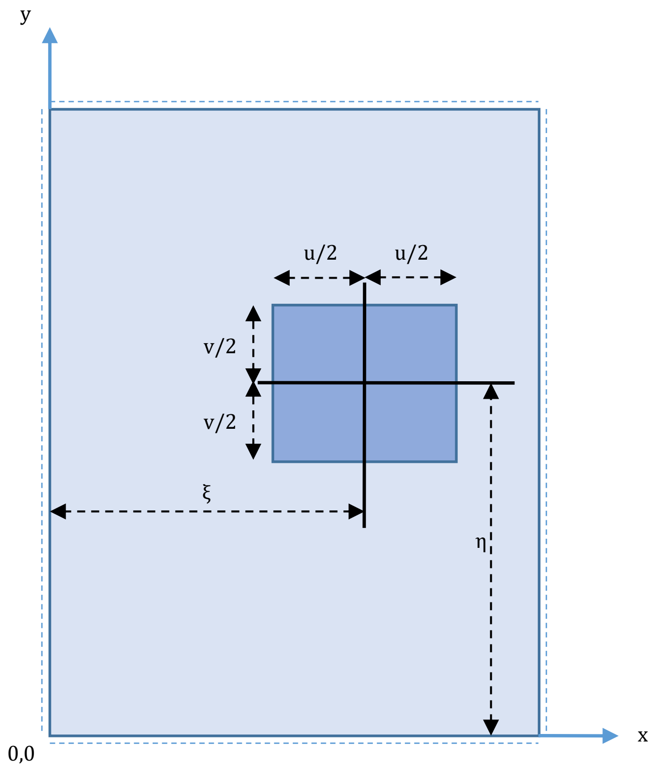
With the computer technology and software development tools that we have available to us, we can compute any number of results and visualise them in any way we want when solving these equations in various ways. The following requirement specifications describe an office automation application in continuous expansion, later used to explore how results can be presented.

Firstly, the requirement for a correct and complete problem definition:
We need to define a parameterised model based on the CPT that includes all the variables that are included in the PDEs:
- the dimensions of the plate, i.e., the width, length and thickness
- the material properties for Hooke's Law: flexural rigidity D represented by Young's modulus and Poisson's ratio
- the load types that can be represented using the Navier's general solution
- the plate thickness may wary in either X- or Y-direction, expressed as a function in thickness (h) or the flexural rigidity (D) of a plate.
- the material properties in CPT assumes an isotropic material, but the PDE can be expanded to include an orthotropic material definition, still following Hooke's Law.

All of the variables are represented by symbols in the CPT equations. By varying all of these variables a user can explore what happens when the variables are specified across discrete values and continuous ranges.

The range of boundary conditions need to be expanded beyond Simply Supported on all four edges, to include all the possible cases that will satisfy the CPT equation, see table 2.

To cover all these aspects of a rectangular plate, we need to expand the solution methods to include both Navier's and Lévy's methods and to make use of the superposition principle that are demonstrated so successfully in /6/ i.e., to combine the results for different load types with bending moments on the respective simply supported boundaries to represent alternative boundary conditions. All these aspects are described in part in /6/ all we need to do is to expand the theoretical material to fill the search space we want to cover.

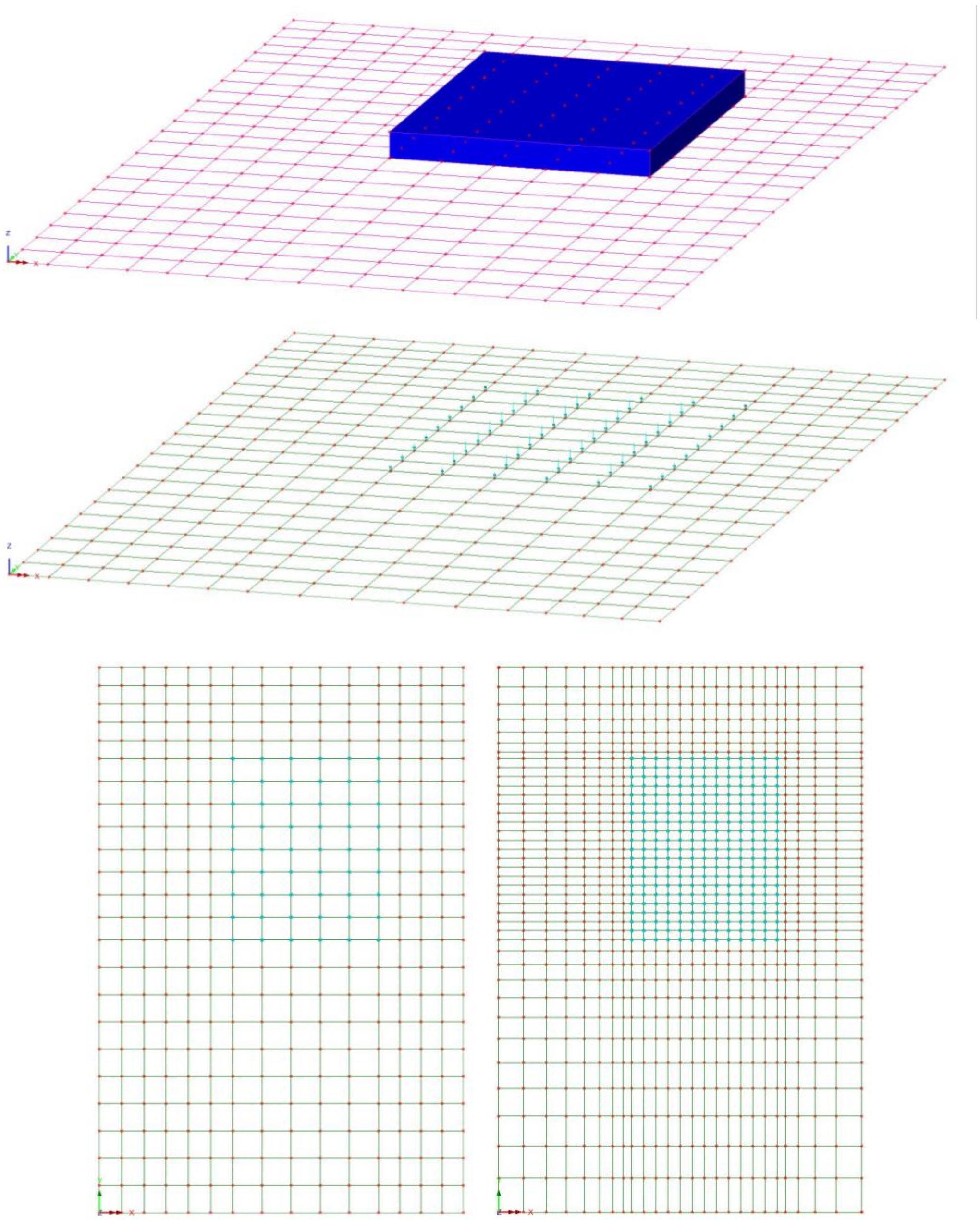


$$w = \frac{1}{\pi^4 D} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \frac{a_{mn}}{\left(\frac{m^2}{a^2} + \frac{n^2}{b^2}\right)^2} sin\frac{m\pi x}{a} sin\frac{n\pi y}{b} \quad (9)$$

Where

$$a_{mn} = \frac{16P}{\pi^2 mnuv} Sin\frac{m\pi\xi}{a} Sin\frac{n\pi\eta}{b} \; Sin\frac{m\pi u}{2a} \; Sin\frac{n\pi v}{2b} \quad (10)$$

Figure 9: The definition of a patch load for Navier’s solution method



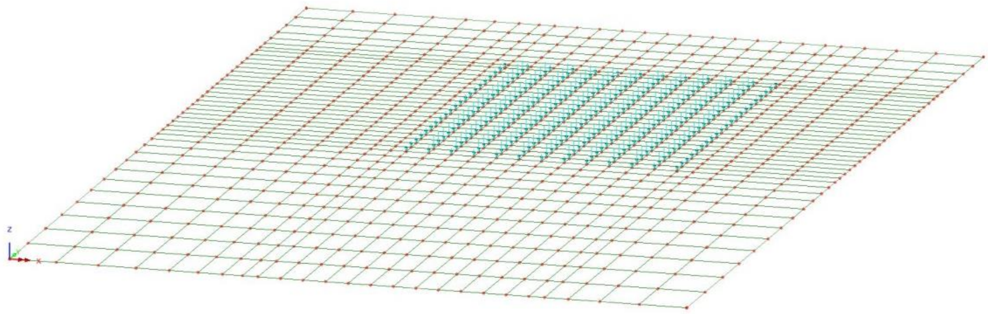


Figure 10: The discretised model for a patch load with moderate biases

Secondly, the requirement for a conversion of the geometry into a discretized model:
The geometry must be discretized into a number of points where the results are to be calculated and then used for visualisation as lines between the points. The natural choice is to define a regular grid of points for calculating the Result Components. For visualisation purposes lines are drawn parallel to the X- axis and Y-axis respectively, and the points can be used as nodes for elements in a numerical analysis.

Each result component is represented as a grid of points with 3D coordinates where X- and Y-coordinates represent the position in the plate and the Z-coordinate the calculated value at that point.

The use of a regular grid allows us to create an identical finite element mesh that can be sent across to an FEA system, by choice LUSAS from FEA Ltd /24/ for a numerical analysis using four-noded thin shell elements. Both four-noded and eight-noded shell elements should be catered for. When we bring the node results back from the FEA analyses, we can display them in the same way as the analytical results and compare them like for like.

The position of the grid points should be controlled by division numbers along the edges and should be varied in four alternative ways:
- The number of lines in the grid can be set to any number using even distance between the lines in X- and Y-directions independently of each other.
- The distance between the lines can be varied in one section in each direction allowing a uniform or a graded mesh across the plate.
- The distance between the lines can be varied in two sections in each direction using two biases in both X- and Y-direction. The bias can be specified in two sections to enable a concentration of calculated points along the boundary on either side for the full load or a concentration anywhere in the plate.
- The distance between lines can be varied in three sections in each direction using three biases in both X- and Y-direction. The bias can be specified in three sections, either side of the patch load and in the flow lines that go underneath it. This enables the concentration around and underneath a patch load and/or along the boundaries of a rectangular plate.

The user should have a high-level control of how the bias should be distributed; the width of the flowlines is computed from the plate dimensions, the position of the patch load and the ratio of change. The position of the patch load can be parameterized, allowing the user to change position and size of the patch load in a sequence of analyses. The bias calculations should follow automatically.

The concept of Results of Interest can be assigned to any of the lines in both X- and Y-directions. With careful choice of division numbers for a sequence of analyses the Result of Interest lines can be placed in the same position for all the analyses, making it possible to directly compare the results in a convergence study for both analytical and numerical analyses. Individual points can be designated as Results of Interest, allowing for example the midpoint to always be in focus, as well as the corners of a patch load, etc.

Thirdly, the requirement for a user interface and the plotting of results:
The search space defined in this way can be implemented in a computer environment we are all familiar with, Microsoft Office. To create the user interface where a user can specify the values and ranges for the variables, and to handle the amount of results produced, it has to be a relational database application using Access and Visual Basic for Applications (VBA).

The results can be plotted as graphs in Excel, but that is too limiting. The alternative is to feed the results back to the LUSAS pre-processor as for display of the plate dimensions plus the points and curves representing the individual Result Component distributed across the plate. The added benefit is that the results can be presented as a model in the pre-processor, allowing the user to manipulate what to include in the presentation and in addition be able to rotate, scale, pan and zoom.
When using the same discretization for both analytical and numerical analyses, nodal results can be compared like for like. A series of discretizations can be shared between the analytical and numerical analyses, enabling a user to study convergence rates for results from the two alternative sources.

This is particularly useful for convergence studies of numerical results, to ensure that the analyses converge at the desired rate in both analytical and numerical results.

Fourthly, the requirement for a range of mathematical models:
The three relationships that combine to form the PDE for the CPT can be changed to well-known alternatives to form alternative PDEs. Figure 6 and table 1 show alternative Formulations across the thickness range.

When alternative components for forming a PDE are brought together, different and more versatile PDEs are formulated. These may better represent the physical problem that the analysis should represent and produce more accurate results.

7. A multi-dimensional search space

All these possibilities combined form a multi-dimensional search space where a user may select which permutation of variables represent their problem most accurately. Each of the dimensions can be represented as a set of discrete variables or a continuous range. An algorithm can be designed to go through a set of cases in turn, producing results for both analytical and numerical analyses. To keep control of the enormous amount of results that can be produced, a relational database is necessary.

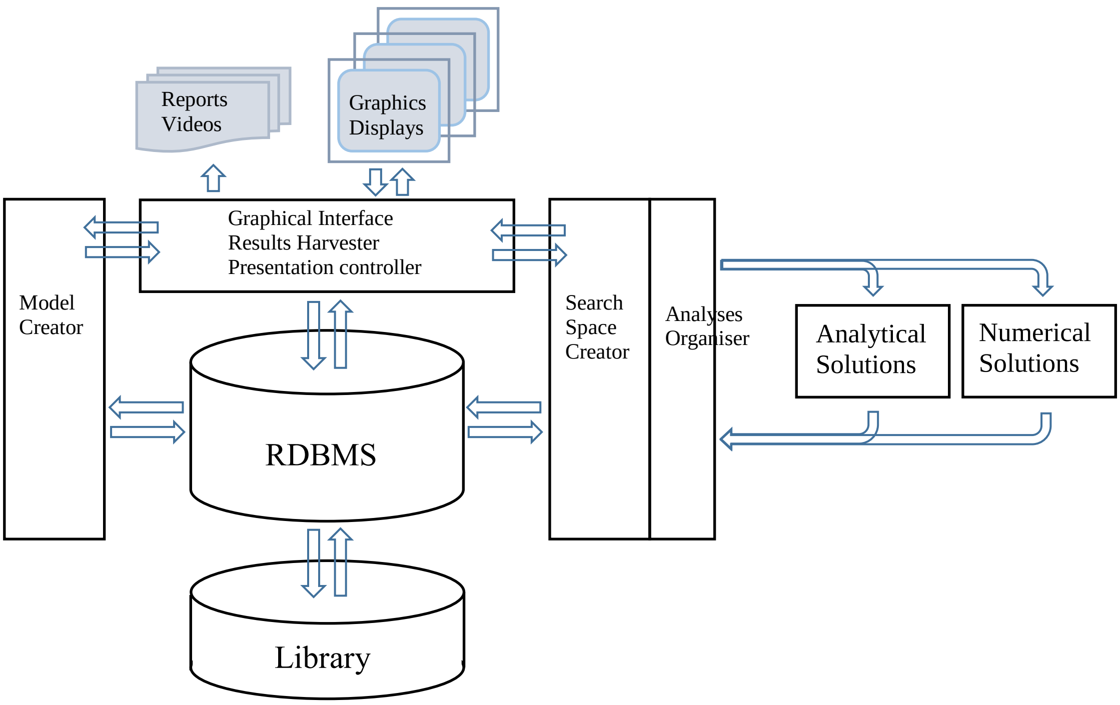


Figure 11: system architecture of an engineering simulator as described

8. What can we do today using modern computer technology?

Today's engineers have access to vast computer power for calculating problems with unlimited complexity and with unlimited means of visualisation of the results. Database technology holding the Results of Interest from hundreds of analyses will enable structural engineers to answer **"What-if?"** questions based on simulation results, analytical and numerical, comparing and extracting detailed information from long sequences of analyses using tools widely used in office automation, such as Pivot Tables /25/ and Analytics /26/.

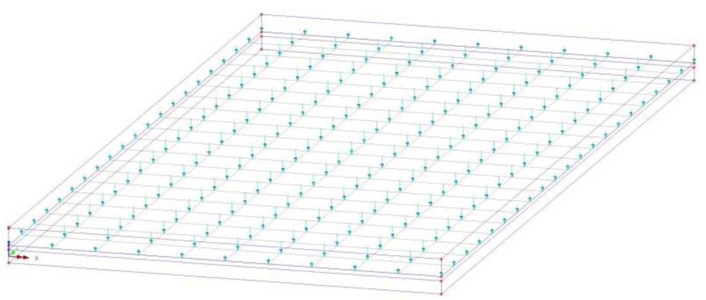
The technologies that will make a difference include:
- Relational Databases and the tools used to create, manipulate and present the content of small (Microsoft Access) and large (SQL Server, Oracle, DB2, etc.)

databases.
- Office Automation tools (Microsoft Office, etc.) widely used across all industries for creation and management of data;
- Visualisation technology where results can be displayed as models in an interactive software environment;
- This will enable the manipulation and creation of sequences of pictures, videos of stored results, interactive exploration of results across many analyses, extraction of results of interest ;
- Software technology to run hundreds of analytical and numerical analyses spanning a multi-dimensional search space;
- Computer hardware to run hundreds of analyses in reasonable clockwall time.
- User interface technology where the software understands intent and can direct the user to information of importance based on high level user input;
- User interface technology where parameters in a search space can be interactively controlled, the users setting fixed values and ranges, for example for shape dimensions, load positions and boundary conditions.
- Artificial Intelligence technology to direct searches in multi-dimensional search spaces to find a group of solutions that meet criteria set up-front.
- And there is more…, much more …

To demonstrate some of the possibilities that we have available to us, a simple, but effective implementation of a rectangular plate with even thickness, all edges simply supported and with uniformly distributed load is used. These examples make use of the functionality of the Access database application detailed previously.

9. An example of what can be done today

Here is a paper presentation of some of the results that can be generated in a computer program outline in the above section.



Lx = 10000 mm
Ly = 20000 mm
H = 300 mm

Grid in X-direction: 10
Grid in Y-direction: 20
Fourier terms: 10 & 10

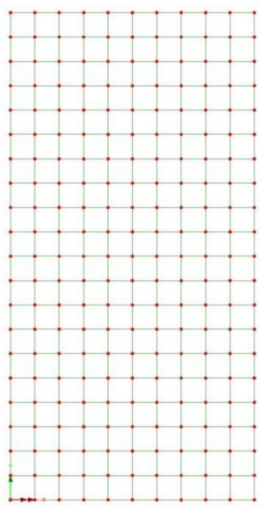


Figure 12: A rectangular plate with a full uniform load and the discretized shape with the calculation points given

The analysis is set up to simulate a reinforced concrete plate with dimensions given as Lx = 10000mm, Ly = 20000mm and thickness = 300mm. The analytical definition and the discretized models are shown in figure 12.

The number of terms in the Fourier series is 10 for both x and y. Some of the results from the analysis are shown in figures 12 and 13.

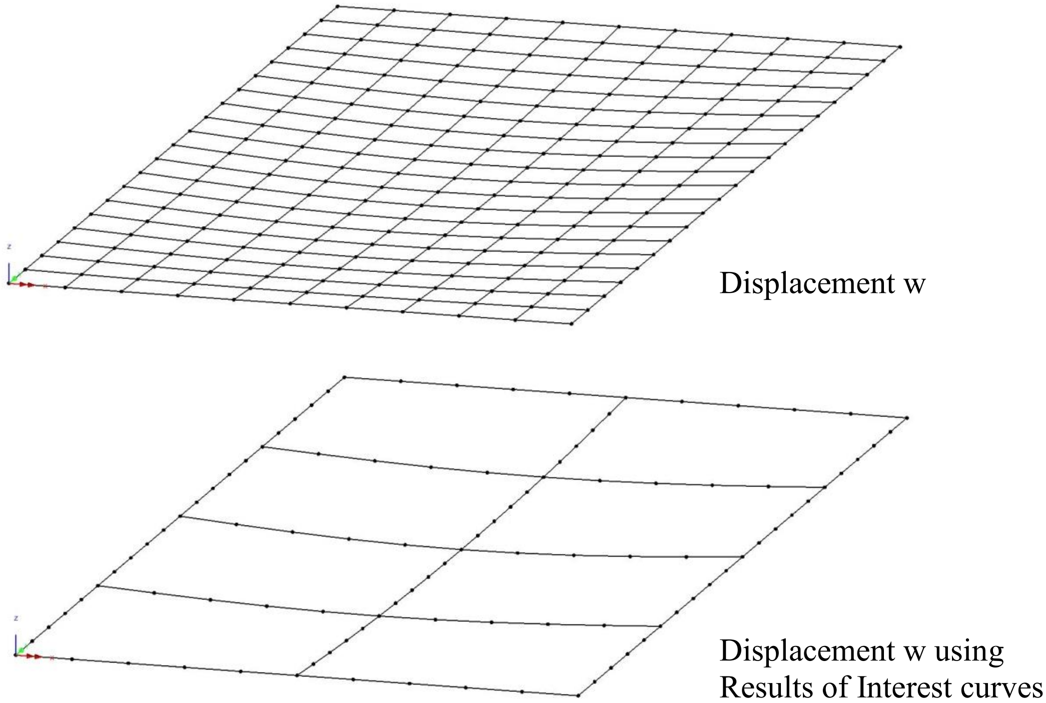
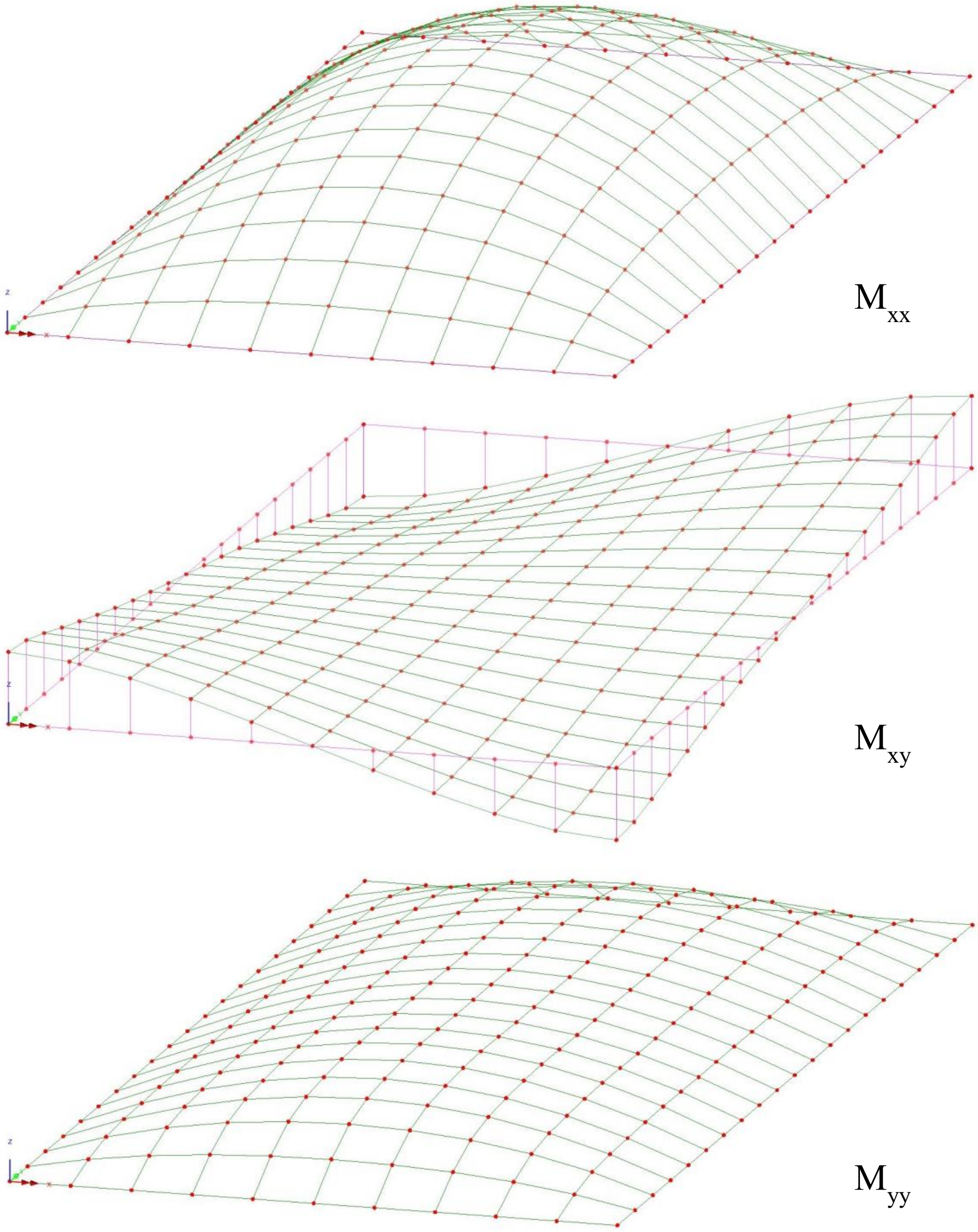


Figure 13: Deformation of the plate in figure 11shown as a grid of curves in X- and Y-directions, top: all curves in the grid; bottom: a selected set of Results of Interest curves,



All ten result components are calculated and can be displayed as models as shown in figure 13 and 14 for the bending moments. These plots are useful for understanding of the qualitative behaviour of rectangular plates. Additional output in terms of numerical results must complement the models for effective quantitative evaluations.

However, plotting the result components as a grid of curves and surfaces on paper doesn’t give them justice, only when the user can work with the results as a model by interactively rotate, scale, pan and zoom will they come to life. This will be demonstrated in the presentation.

10. From the first computer results to continuous display

Modern computer technology can revive the classic material in a way never dreamed of by the originators. Not only to replicate what they did, i.e., compute the same tables again, but to create results as shapes in an FEA pre-processor for a user to select how the results should be presented and then view the model from any angle. A video sequence can be created for a set of related result models. These can be put together from the same analysis or from several analyses, making it possible to see how a plate behaves in a systematic variation of input data across a sequence of analyses. With a unified storage format for analytical and numerical results, it will be possible to combine them in the same video sequences, all being a replay of already stored data.

In the Access office automation application specified here, the structure of the result model is identical for all the Result Components, the only difference between them is the Z-coordinate values. Hence, by feeding a series of point coordinate sets in succession into a shared structure for the model, any number of changes in the underlying data can be displayed in a live sequence. This way we can create a continuous change of the results, i.e., a live video as in ParaView /27/:

- all 10 result components can be displayed in a sequence
- variations in for example any of the bending moments can be shown for variation in a/b ratios, thickness, load-types, boundary conditions.
- variation in response for a sequence of numerical results can be compared side-by-side with the analytical results, for example convergence studies
- variations in response to plate thickness changes across the different Formulations

The user interface in the FEA pre-processor is customized to allow roll forward and backward through the data sets, so a user can clearly view the changes that happens between the data sets again. The history of the sequences can be stored as a video sequence for later replay.

Beyond this, we want to display live the data as they are created, a window into the calculation processes as they happen. This requires integration of analysis tools, database technology, graphics display and High-Performance Computing (HPC). The software used in the demonstrations is a step in the right direction.

However, this is the first three rungs on a long ladder. The next rung must allow a user to display several streams of data on individual displays and to control the calculation process using a games controller.

Rectangular plates modelled using CPT are just the start. The key question is**:** **"What problem would you like to solve with this technology at your fingertips?"**

11. Summary

The engineering content in this presentation may seem simple, but problem complexity is not the point of the presentation, rather than to explore the opportunities that open up when exploiting well-established computer technology to engineering analysis. The ideas outlined here are implemented in an experimental Access application and is “work in progress”. So far, it uses well-established computer hardware and software found on any desk in any engineering analysis company.

Some examples are included in the paper and presentation, but many more examples can be shown to demonstrate how interactive visualisation of results can enhance the understanding of how any structure behave under load. The presentation of the material covered in this paper comes in four parts:
- this paper that outlines the ideas behind the engineering workbench, stating with rectangular plates;
- the PowerPoint presentation used at the conference presentation;
- the live demonstration of how analysis results can be presented in an FEA pre-processor;
- a library of videos that make the ideas described here come to life.

The bi-harmonic forth-order CPT partial differential equation is one of many PDEs that can be revisited to create a comprehensive library of analytical solutions. The material exists in an incomplete form, it is a large jig-saw puzzle that needs to be put together in a systematic way and offered as a simulation workbench. A project to significantly expand the range of analytical solutions for use as the exact solution in Formulation, Verification and Validation is well overdue.

The additional purpose for such a development is to meet the learning and teaching requirements found among members of the following groups:
- practicing engineers in structural engineering,
- lecturers at universities teaching the engineering students the plate and shell theory and everything beyond;
- the new generation of computer games savvy youngsters who should be encouraged to choose a career in design and structural engineering.

The engineering workbench will be a key recruitment tool for the new generation of youngsters to choose a career in mathematical modelling. They will be lost to other and more exciting careers if they cannot work with the computer tools they have grown up with and enjoy their time at work playing games called mathematical modelling.

12. Conclusions

We owe it to the originators of the underlying mathematics for Classical Solid Mechanics to demonstrate what they could have done themselves if they only had the same computer technology as we have. There are no limits to the opportunities available today to advance the understanding of mathematical models, i.e., this is a Blue-Sky Opportunity.

What is explored here is what we currently can do in June 2019. However, computer technology is moving forward at an accelerating speed. Using computers the same way as in the 1960s and 1970s doesn't measure up. We must use computers the way the computer game generation is accustomed to. We must move faster than in the last two decades to keep up with the pace of change, let alone catch up on the advances made. Established thinking and working practices must be replaced by new paradigms for shape representation /28/ and new software development in user interfaces, computer graphics and multi-computer job administration, in the use of relational databases, application scripting and most of all in exploiting the advances in High Performance Computing (HPC).

As an example, the first Exa Floating Point Operations Per Second (FLOPS) computers are predicted available in 2021 /29/ computers that can do 1,000,000,000,000,000,000 FLOPS. To exploit the enormous computer power soon available at affordable prices, the established practices of **"ONE big analysis"** have to be replaced by **"MANY smaller analyses"** in Formulation, Validation & Verification based working practices, combining analytical and numerical analyses. Scaling and tuning of the technology will in time create **"MANY big analyses"**. Many-core heterogeneous parallel computers will open up for parallel processing at an unimaginable scale, compressing the clock wall time so simulations that took weeks instead become interactive. **What do you plan to do today so you are ready when Exa FLOPS machines arrive?**

I for one have ambitions in this respect: I want Pica FLOPS, or better, Exa FLOPS, computer power hanging off the tip of my games controller, with the software functionality to compute, capture and present the results coming out of the multi-simulations in real-time. Most of the computer technology is here already, **when will software for mathematical modelling catch up?**

13. References

There is an unending list of possible references for the topics covered in this paper. Instead of selecting an arbitrary reference in a long list of possibilities, here is a modern way of directing readers to authoritative sources for further reading: to use the Internet, more specifically Wikipedia as the knowledge resource. Wikipedia gives a clear and concise presentation of most topics with a list of references for further reading.

/1/ about Lagrange: https://en.wikipedia.org/wiki/Joseph-Louis_Lagrange

/2/ about Augustin-Louis Cauchy: https://en.wikipedia.org/wiki/Augustin-Louis_Cauchy

/3/ about Claude-Louis Navier: https://en.wikipedia.org/wiki/Claude-Louis_Navier

/4/ Fung, Y. C., Tong, Pin, Chen Xiaohong (2017) “Classical and Computational Solid Mechanics”, Second Edition, World Scientific Publishing Co. Pty. Ltd.

/5/ Reddy, J. N. (2007), “Theory and Analysis of Elastic Plates and Shells”, Second Edition, CRC Press, Boca Raton, FL, USA,Chapter 3, p95-123.

/6/ Timoshenko, Stephen P. and Woinowsky-Krieger, S. “Theory of Plate and Shells”, Second Edition, McGraw-Hill, article 28, pp 108-110.

/7/ Bittner, Ernst: (first edition 1938, second edition 1965) “Platten und Behälter”, Springer Verlag, Wien.

/8/ Rűdiger-Urban (1955), “KreisZylinderscalen - Circular Cylindrical Shells”, B. G. Teubner Verlagsgesellschaft Leipsig.

/9/ About Kelvin Functions: https://en.wikipedia.org/wiki/Kelvin_functions

/10/ Holand, Ivar, (1961), “Tables for the Analysis of Cylindrical Tanks or Tubes with Linearly Variable Thickness”, International Association for Bridge and Structural Engineering, Zurich.

/11/ Gibson, J. E. (1954), “The Design of Cylindrical Shell Roofs”, E & F N Spon Ltd., London

/12/ Gibson, J. E. (1960), “The Design of Cylindrical Shell Roofs”, E & F N Spon Ltd., London

/13/ Gibson, J. E. (1975), “Computing in Structural Engineering”, Applied Science Publishers Ltd, London

/14/ About Fortran: https://en.wikipedia.org/wiki/Fortran

/15/ Timoshenko, Stephen P. and Woinowsky-Krieger, S. “Theory of Plate and Shells”, Second Edition, McGraw-Hill.

/16/ Flűgge, Wilhelm (1960), “Stresses in Shells”, Third Printing, Springer-Verlag New York Inc.

/17/ Scabó, Barna and Babuška, Ivo, (2011), “Introduction to Finite Element Analysis, Formulation, Verification and Validation”, John Wiley & Sons, Ltd, UK

/18/ ASME V&V 10-2006, Reaffirmed 2016, “Guide for Verification and Validation in Computational Solid Mechanics”, The American Society of Mechanical Engineers, New York.

/19/ ASME V&V 10.1-2012, "An Illustration of the Concepts of Verification and Validation in Computational Solid Mechanics", The American Society of Mechanical Engineers, New York.

/20/ Blezinger, Kai-Uwe, (2008) "Theory of Plates, Part 2: Plates in Bending", Lecture Notes, Winter Semester 2012/2013, Lehrstuhl Fur Statik, Technische Universitat Munchen, Deutschland, p8.

/21/ Roarche, Patrick J. (2009), "Fundamentals of Verification and Validation", Hermosa Publishers, Albuquerque, NM.

/22/ Timoshenko, Stephen P. and Woinowsky-Krieger, S. "Theory of Plate and Shells", Second Edition, McGraw-Hill, Article 22, p83-87.

/23/ Timoshenko, Stephen P. and Woinowsky-Krieger, S. "Theory of Plate and Shells", Second Edition, McGraw-Hill, Chapter 7, Articles 52-56, p229-258

/24/ Lusas: http://www.lusas.com/

/25/ About Pivot Tables: https://en.wikipedia.org/wiki/Pivot_table

/26/ About Analytics: https://en.wikipedia.org/wiki/Analytics

/27/ About ParaView: https://en.wikipedia.org/wiki/ParaView and ParaView official site: https://www.paraview.org/

/28/ Aas, Jon (2019), "Dislocation Meshing, a Credible Solution to Automatic Hexahedral Meshing", NAFEMS World Congress & SPDM Conference 2019, Quebec, Canada.

/29/ about Exa FLOPS computing: https://en.wikipedia.org/wiki/Exascale_computing